Line bundles on curves

$k = \bar{k}$

A curve is a connected one dim'l variety $/ k = \bar{k}$

notation: $h^i(\mathcal{F}) := h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$

$|X|$ = set of closed pts

Def The (arithmetic) genus of a proper curve X is the (nonnegative) integer

$$g := g(X) := \dim_k H^1(X, \mathcal{O}_X)$$

Recall that divisors on curves are formal $\mathbb{Z}$-linear combinations of closed points

Def X curve

$$\deg : \operatorname{Div}(X) \to \mathbb{Z}$$
$$\sum n_i p_i \mapsto \sum n_i$$

Theorem (Riemann–Roch, first form)

X ns proper curve, $D \in \operatorname{Div}(X)$

$$\text{(RR)} \quad h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = 1 - g + \deg(D)$$

<u>Proof</u> if $D = 0$ RR by def.

Use s.e.s's

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \to \mathcal{O}_p \to 0$$

$\Rightarrow$ l.e.s.

$$\begin{array}{l} 0 \to H^0(\mathcal{O}_X(D-p)) \to H^0(\mathcal{O}_X(D)) \to \overset{k}{\overset{\parallel}{H^0(\mathcal{O}_p)}} \to \\ \to H^1(\mathcal{O}_X(D-p)) \to H^1(\mathcal{O}_X(D)) \to \underset{0}{\underset{\parallel}{H^1(\mathcal{O}_p)}} \to 0 \end{array}$$

$\Rightarrow$
$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D-p)) - h^1(\mathcal{O}_X(D-p)) + 1$$

$\Rightarrow$ RR' for D $\Leftrightarrow$ RR' $D-p$ $\square$

<u>Cor.</u> if $D \sim D'$, then $\deg(D) = \deg(D')$

<u>Def</u> $\mathcal{L}$ line bundle on X

$$\deg(\mathcal{L}) = \deg(\mathrm{div}(s))$$

for any $s \in \mathcal{L}_\eta$

<u>Cor</u> (Riemann-Roch, second form)

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^\vee \otimes \omega_X) = 1 - g + \deg(\mathcal{L})$$

Lemma (1) $\deg(\mathcal{L}) < 0 \Rightarrow h^0(X, \mathcal{L}) = 0$

(2) $\deg(\mathcal{L}) = 0 \Rightarrow h^0(X, \mathcal{L}) = 0, 1$

$\deg(\mathcal{L}) = 0,\ h^0(X, \mathcal{L}) = 1 \Leftrightarrow \mathcal{L} \cong \mathcal{O}_X$

(3) $\deg(\mathcal{L}) > 2g-2 \Rightarrow h^0(X, \mathcal{L}) = 1 - g + \deg(\mathcal{L})$

(4) $p_1, \dots, p_n \in |X|$ pairwise distinct

$\Rightarrow h^0(\mathcal{L}) - h^0(\mathcal{L}(-\sum p_i)) \leq n$

Proof $h^0(\mathcal{L}) > 0 \Leftrightarrow \exists s \in \Gamma(X, \mathcal{L}) \setminus \{0\}$

$\Leftrightarrow \exists s \in \mathcal{L}_\eta \setminus \{0\}$ s.t. $\mathrm{div}(s)$ effective.

$h^0(\mathcal{L}) > 0 \Rightarrow \deg(\mathcal{L}) = \deg(\underbrace{\mathrm{div}(s)}_{\text{effective.}}) \geq 0$

$\Rightarrow$ (1) ✓

$\mathrm{div}(s)$ effective, $\deg(\mathrm{div}(s)) = 0$

$\Rightarrow \mathrm{div}(s) = 0$ & $\mathcal{L} \cong \mathcal{O}_X(\mathrm{div}(s)) \cong \mathcal{O}_X$

$\Rightarrow$ (2)

$\deg(\mathcal{L}) > 2g-2 \Rightarrow \deg(\mathcal{L}^\vee \otimes \omega_X)$

(1) $\Rightarrow h^0(\mathcal{L}^\vee \otimes \omega_X) = 0$

RR $\Rightarrow h^0(\mathcal{L}) = 1 - g + \deg(\mathcal{L})$

l.e.s. of $0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{O}_p \to 0$
+ induction

$\rightsquigarrow$ (4)

$\square$

Lemma 1) if for all $p \in |X|$ $h^0(\mathcal{L}) - h^0(\mathcal{L}(-p)) = 1$

then $\mathcal{L}$ is globally generated a.k.a. base pt free.

2) if for all $p, q \in |X|$ distinct

we have $h^0(\mathcal{L}) - h^0(\mathcal{L}(-p-q)) = 2$

then the linear system $|\mathcal{L}|$ separates points (i.e. $X \to \mathbb{P}(H^0(\mathcal{L}))$ is injective on closed points)

3) if for all $p \in |X|$ we have $h^0(\mathcal{L}) - h^0(\mathcal{L}(-2p)) = 2$

then the linear system $|\mathcal{L}|$ separates tangent vectors (i.e. $X \to \mathbb{P}(H^0(\mathcal{L}))$ is injective on tangent spaces)

Pf $H^0(\mathcal{L}(-p))$ = sections of $\mathcal{L}$ that vanish @ p

$$0 \to H^0(\mathcal{L}(-p)) \to H^0(\mathcal{L}) \xrightarrow{res} H^0(\mathcal{L}|_p) \to$$

if $h^0(\mathcal{L}) - h^0(\mathcal{L}(-p)) = 1$ for all p

$\Rightarrow$ res surjective $\Rightarrow$ $\mathcal{L}$ base pt free

$\exists$ section of $\mathcal{L}$ that vanish @ p but not at q and vice versa

if $h^0(\mathcal{L}) - h^0(\mathcal{L}(-p-q)) = 2$ for all p, q distinct

then

$$\begin{tikzcd}
& p \mapsto \mathbb{P}(H^0(\mathcal{L}(-q))) \arrow[dr] & \\
X \arrow[ur] \arrow[rr, "\pi"] \arrow[dr] & & \mathbb{P}(H^0(\mathcal{L})) \\
& q \mapsto \mathbb{P}(H^0(\mathcal{L}(-p))) \arrow[ur] &
\end{tikzcd}$$

$\Rightarrow$

$$\pi(p) \notin \mathbb{P}(H^0(\mathcal{L}(-p))) \ni \pi(q)$$
$$\pi(p) \in \mathbb{P}(H^0(\mathcal{L}(-q))) \not\ni \pi(q)$$

For the last point: want to see that

$$\mathfrak{m}_{\mathbb{P},\pi(p)} / \mathfrak{m}^2_{\mathbb{P},\pi(p)} \longrightarrow \underbrace{\mathfrak{m}_{X,p} / \mathfrak{m}^2_{X,p}}_{\text{one-dim'l.}}$$

is surjective

"there is a section of $\mathcal{L}$ which vanishes @ p with order 1 but not order 2

$$0 \to H^0(\mathcal{L}(-p)) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}|_p)$$

$$0 \to H^0(\mathcal{L}(-2p)) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}_p \otimes \mathcal{O}_{X,p}/\mathfrak{m}^2)$$

$$\pi^\flat : \mathcal{O}_{\mathbb{P},\pi(p)} \to \mathcal{O}_{X,p}$$

$$\frac{x_i}{x_0} \longmapsto \frac{s_i}{s_0}$$

$$\exists \; s \in H^0(\mathcal{L}(-p)) \setminus H^0(\mathcal{L}(-2p)) \subseteq H^0(\mathcal{L})$$

$$s = \sum a_i s_i \quad a_i \in k$$

$$\Rightarrow \pi^\flat\Big(\sum_i a_i \frac{x_i}{x_0}\Big) = \frac{s}{s_0} \in \mathfrak{m}_X \setminus \mathfrak{m}_X^2 \qquad \square.$$

<u>Theorem</u> $\pi: X \to Y$ projective morphism of finite type k-schemes

π is a closed embedding iff

(1) $\pi: X(k) \to Y(k)$ is injective

(2) $d\pi_x : (m_{X,x}/m_{X,x}^2)^\vee \to (m_{Y,\pi(x)}/m_{Y,\pi(x)}^2)^\vee$ is injective for all $x \in X(k)$

<u>Proof</u>

<u>Def</u>: $\phi: X \to Y$ finite morphism.

for all $q \in Y$ define.

$$\deg(\phi, q) := \operatorname{rank}(\underbrace{\phi_* \mathcal{O}_X|_q}_{=(\phi_* \mathcal{O}_X)_q \otimes k_Y(q)})$$

<u>Recall</u> $\mathcal{F}$ coherent sheaf on Y

then $q \longmapsto \operatorname{rank}(\mathcal{F}|_q)$ is

upper-semicontinuous.

$\Leftrightarrow$ $\forall q \in \overline{\{q'\}}$ $\operatorname{rank}(\mathcal{F}|_q) \geq \operatorname{rank}(\mathcal{F}|_{q'})$

Y noetherian

<u>Step 1</u> $\varphi: X \to Y$ finite s.t. for all $q \in Y$ $\deg(\varphi, q) \in \{0, 1\}$

$\Rightarrow$ φ is a closed embedding.

question is local on target:

assume wlog $Y = \mathrm{Spec}(B)$

ϕ finite $\Rightarrow$ ϕ affine $\Rightarrow X = \mathrm{Spec}(A)$

$\phi^b : B \longrightarrow A$ makes A a finite

B-module.

Consider exact sequence of B-mod.

$$B \longrightarrow A \longrightarrow N := \mathrm{coker}_B(\phi) \longrightarrow 0$$

$\Rightarrow$ $\forall\, m \triangleleft B$ max'l have exact

sequence

$$B \otimes_B k_B(m) \xrightarrow{\varphi_m^b} A \otimes_B k_B(m) \rightarrow N \otimes_B k_B(m) \rightarrow 0$$

$$\dim_{k_B(m)}\left(A \otimes_B k_B(m)\right) = \deg(\phi, m) \in \{0,1\}$$

$$\Rightarrow \left\{\begin{matrix} A \otimes_B k_B(m) = 0 \\ \text{or} \\ \varphi_m^b \text{ iso} \end{matrix}\right\} \Rightarrow \begin{matrix} N \otimes_B k_B(m) = 0 \\ \text{for all } m \in B \end{matrix}$$

Nakayama's lemma $\Rightarrow N = 0$

// Step 1

Step 2 reduction of theorem to Step 1.

question local on target

assume wlog $Y = \mathrm{Spec}(B)$

Upper semicontinuity of fibre dimension

$\Rightarrow Y_{\geq 1} = \{ q \in Y \mid \dim(\phi^{-1}(q)) \geq 1 \} \subseteq Y$

is a closed subset.

(1) $\Rightarrow$ $Y_{\geq 1} = \emptyset$

$\left.\begin{array}{l}\Rightarrow \phi \text{ quasi-finite} \\ \phi \text{ projective}\end{array}\right\} \Rightarrow \phi$ finite

$q \mapsto \deg(\phi, q)$ upper semicontinuous.
$\deg(\phi, q) = 1$ for all closed q
$\Rightarrow$ ϕ is as in Step 1. $\square$

Exercise prove that closed embeddings are injective on k-points and injective on tangent spaces.

<u>Exercise</u> (1) $\deg(\mathcal{L}) \geq 2g \Rightarrow \mathcal{L}$ globally gen.
(2) $\deg(\mathcal{L}) \geq 2g+1 \Rightarrow \mathcal{L}$ very ample
(3) $\deg(\mathcal{L}) > 0 \Rightarrow \mathcal{L}$ ample.

(Practice in definitions)

<u>Cor</u> X ns proper curve $\Rightarrow$ X projective.
<u>Pf</u> $p \in |X|$ $\deg(\mathcal{O}_X(p)) = 1$
$\Rightarrow \mathcal{O}_X(p)$ ample $\square$

Cor: X ns proper curve $/ k = \bar{k}$

$g(X) = 0 \Rightarrow X \cong \mathbb{P}^1$

Pf $p \in |X|$ $\deg(\mathcal{O}_X(p)) = 1 \geq 2g(X) - 1$

$\Rightarrow \mathcal{O}_X(p)$ very ample

$h^0(\mathcal{O}_X(p)) = 1 - 0 + \deg(\mathcal{O}_X(p)) = 2$

$\Rightarrow X \xhookrightarrow{i} \mathbb{P}(H^0(\mathcal{O}_X(p))) = \mathbb{P}^1$

X proper $\Rightarrow$ i isomorphism $\square$

Analysis for genus 2 maps:

X ns proper curve of genus g.

$$\left.\begin{array}{l} \deg(\omega_X) = 2 \\ h^0(\omega_X) = 2 \end{array}\right\} \Rightarrow \omega_X \text{ base point free}$$

$\rightsquigarrow X \xrightarrow{\pi} \mathbb{P}(H^0(\omega_X)) \cong \mathbb{P}^1$

$2 = \deg(\omega_X) = \deg(\pi^* \mathcal{O}(1)) = \deg(\pi)$

$\left\{ \text{genus 2 } \underset{2:1}{\text{covers}} \text{ of } \mathbb{P}^1 \Big/ \mathrm{Aut}(\mathbb{P}^1) \right\} \longleftrightarrow \text{genus 2 curves}$